

ON THE FI-MODULE STRUCTURE OF $H^i(\Gamma_{n,s})$

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ABSTRACT. The groups $\Gamma_{n,s}$ are defined in terms of homotopy equivalences of certain graphs, and are natural generalisations of $\text{Out}(F_n)$ and $\text{Aut}(F_n)$. They have appeared frequently in the study of free group automorphisms, for example in proofs of homological stability in [8, 9] and in the proof that $\text{Out}(F_n)$ is a virtual duality group in [1]. More recently, in [5], their cohomology $H^i(\Gamma_{n,s})$, over a field of characteristic zero, was computed in ranks $n = 1, 2$ giving new constructions of unstable homology classes of $\text{Out}(F_n)$ and $\text{Aut}(F_n)$. In this paper we show that, for fixed i and n , this cohomology $H^i(\Gamma_{n,s})$ forms a finitely generated FI-module of stability degree n and weight i , as defined by Church-Ellenberg-Farb in [2]. We thus recover that for all i and n , the sequences $\{H^i(\Gamma_{n,s})\}_{s \geq 0}$ satisfy representation stability, but with an improved stable range of $s \geq i + n$ which agrees with the low dimensional calculations made in [5]. Another important consequence of this FI-module structure is the existence of character polynomials which determine the character of the \mathfrak{S}_s -module $H^i(\Gamma_{n,s})$ for all $s \geq i + n$. In particular this implies that, for fixed i and n , the dimension of $H^i(\Gamma_{n,s})$, is given by a polynomial in s for all $s \geq i + n$. We compute explicit examples of such character polynomials to demonstrate this phenomenon.

1. INTRODUCTION

It is well known that the group of outer automorphisms of the free group of rank n can be described as the space of self-homotopy equivalences of a graph X_n of rank n , up to homotopy, i.e.,

$$\text{Out}(F_n) \cong \pi_0(HE(X_n)).$$

Similarly the full group of automorphisms of the free group of rank n is the space of homotopy equivalences of a graph $X_{n,1}$ of rank n with a distinguished basepoint ∂ , up to homotopy,

$$\text{Aut}(F_n) \cong \pi_0(HE(X_{n,1})),$$

where homotopies are required to fix the basepoint throughout.

There is a natural generalisation then, where we let $X_{n,s}$ be a graph, by which we mean a connected finite 1-dimensional CW-complex, of rank n with s marked points $\partial = \{x_1, \dots, x_s\}$. We should then consider the group of self-homotopy equivalences of $X_{n,s}$ fixing ∂ pointwise, modulo homotopies through such maps, i.e.,

$$\Gamma_{n,s} := \pi_0(HE(X_{n,s})).$$

In this paper we study the structure of the cohomology $H^i(\Gamma_{n,s})$, always over a field of characteristic zero, as a sequence of \mathfrak{S}_s -modules. The symmetric group \mathfrak{S}_s

1991 *Mathematics Subject Classification.* 20J06, 20F28.

Key words and phrases. FI-modules, Representation stability, Automorphism groups of free groups.

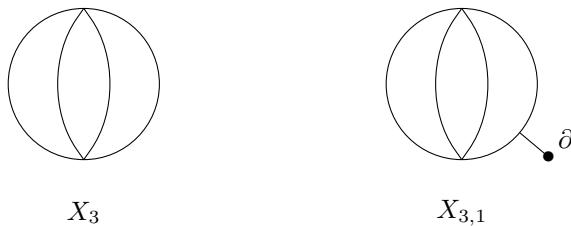


FIGURE 1. Examples of rank 3 graphs that can be used to define $\text{Out}(F_3)$ and $\text{Aut}(F_3)$.

acts on $H^i(\Gamma_{n,s})$ as follows. A homotopy equivalence $h : X_{n,s} \rightarrow X_{n,s}$ permuting ∂ induces an automorphism of $\Gamma_{n,s}$ by conjugation. This automorphism depends, *a priori*, on the choice of h , however, on the level of cohomology it depends only on the permutation. Indeed, if h fixes ∂ pointwise then the induced automorphism is inner, and thus induces the identity on cohomology.

The groups $\Gamma_{n,s}$ have been used, for example, to show that $\text{Out}(F_n)$ and $\text{Aut}(F_n)$ satisfy homological stability in [8, 9], and they appeared in [1] in the proof that $\text{Out}(F_n)$ is a virtual duality group. More recently they were used in [5] to investigate the so called unstable cohomology of $\text{Out}(F_n)$ and $\text{Aut}(F_n)$ by means of an ‘assembly map’

$$H^i(\Gamma_{n_1,s_1}) \otimes \cdots \otimes H^i(\Gamma_{n_k,s_k}) \rightarrow H^i(\Gamma_{n,s}).$$

In particular, in [5] they compute $H^i(\Gamma_{n,s})$ as an \mathfrak{S}_s -module for rank $n = 1, 2$ and use these computations to assemble homology classes in the unstable range of $\text{Out}(F_n)$ and $\text{Aut}(F_n)$. Moreover these computations show that, in rank $n = 1, 2$ and for fixed $i \geq 0$ the sequence $\{H^i(\Gamma_{n,s})\}_{s \geq 0}$ satisfies representation stability; a representation theoretic analogue of homological stability defined by Thomas Church and Benson Farb in [4] (see Definition 2.2). In [5] they use an alternate description of $\Gamma_{n,s}$ as a quotient of a certain mapping class group of a three-manifold, together with general results about representation stability of mapping class groups, to deduce that for any fixed i and n the groups $H^i(\Gamma_{n,s})$ satisfy representation stability with stable range $s \geq 3i$. However, the calculations made in [5] in rank $n = 1, 2$ actually adhere to a bound of $s \geq i + n$. In this paper we improve the stable range to agree with these low rank calculations.

Theorem A. *For fixed i and n the sequence $H^i(\Gamma_{n,s})$ is uniformly representation stable as $s \rightarrow \infty$ with stable range $s \geq n + i$.*

We show this by exhibiting that $H^i(\Gamma_{n,s})$ defines an FI-module (see Definition 2.3). Building on their work in [4], and together with Jordan Ellenberg and Rohit Nagpal, the theory of FI-modules was developed [2, 3], facilitating the application of homological techniques to sequences of \mathfrak{S}_s -modules. We use these techniques to prove the following theorem.

Theorem B. *$H^i(\Gamma_{n,\bullet})$ is a finitely generated FI-module of stability degree n and weight i .*

An important feature of finitely generated FI-modules is the existence of character polynomials; integer-valued polynomials in $\mathbb{Q}[X_1, X_2, \dots]$ where $X_i : \mathfrak{S}_s \rightarrow \mathbb{N}$ is the class function that counts the number of i -cycles. Let $\chi_{H^i(\Gamma_{n,s})}$ denote the character of the \mathfrak{S}_s -module $H^i(\Gamma_{n,s})$.

Corollary 1.1. *There exists a character polynomial $f \in \mathbb{Q}[X_1, \dots, X_i]$ depending on i and n such that for all $s \geq i + n$ and all $\sigma \in \mathfrak{S}_s$,*

$$\chi_{H^i(\Gamma_{n,s})}(\sigma) = f(\sigma).$$

In particular, the dimension of $H^i(\Gamma_{n,s})$ is given by the polynomial $f(s, 0, \dots, 0)$.

One consequence of this result is that, for s sufficiently large, the character $\chi_{i,n}$ is insensitive to cycles of length greater than i . We highlight this phenomenon by computing examples of these stable character polynomials in Section 2.2.

Theorem A and Corollary 1.1 follow immediately from Theorem B by results of Church-Ellenberg-Farb in [2] that we state below (Proposition 2.7, Proposition 2.8). In Section 2 we recall the basic definitions and properties of FI-modules which are then used in Section 3 to show that $H^i(\Gamma_{n,s})$ forms an FI-module. In Section 4 we use a Leray-Serre spectral sequence to prove Theorem B.

Acknowledgements. The author thanks his thesis advisor Martin Kassabov for his insight and for many helpful conversations.

2. REPRESENTATION STABILITY AND FI-MODULES

We work throughout over a field \mathbb{k} of characteristic 0. Let \mathfrak{S}_s denote the symmetric group on s letters. Recall that the irreducible representations of \mathfrak{S}_s correspond to partitions λ of s , denoted $\lambda \vdash s$, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and $\sum_i \lambda_i = s =: |\lambda|$. We denote the irreducible representation corresponding to $\lambda \vdash s$ by P_λ . Let $\lambda \vdash q$. It will be useful to denote by $\lambda[s]$ the ‘padded’ partition $(s - |\lambda|, \lambda_1, \lambda_2, \dots)$ and by $P_{\lambda[s]} = P(\lambda)_s$ its corresponding irreducible representation.

Let P be a \mathfrak{S}_a -module, and Q an \mathfrak{S}_b -module, then we denote the induced representation by

$$P \circ Q = \text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_{a+b}} P \otimes Q$$

(see, for example, Fulton-Harris [6]).

Also, following [5], we denote by $V^{\wedge k}$ the \mathfrak{S}_k -module which is isomorphic as a vector space to $V^{\otimes k}$ where \mathfrak{S}_k acts by permuting the factors and multiplying by the sign of the permutation. That is, let $\text{alt} = P_{(1^k)}$ denote the alternating representation, then

$$V^{\wedge k} = V^{\otimes k} \otimes \text{alt}.$$

For convenience, we recall from [2] the definitions of representation stability and of FI-modules.

Definition 2.1. A sequence $\{V_s\}$ of finite dimensional \mathfrak{S}_s -modules together with linear maps $\phi_s : V_s \rightarrow V_{s+1}$ is called **consistent** if for each s and each $g \in \mathfrak{S}_s$ the following diagram commutes:

$$\begin{array}{ccc} V_s & \xrightarrow{\phi_s} & V_{s+1} \\ g \downarrow & & \downarrow i(g) \\ V_s & \xrightarrow{\phi_s} & V_{s+1} \end{array}$$

where $i : \mathfrak{S}_s \rightarrow \mathfrak{S}_{s+1}$ is the standard inclusion.

Definition 2.2. A consistent sequence $\{V_s\}$ is called **representation stable** if for sufficiently large s each of the following holds.

- (1) (Injectivity) The maps $\phi_s : V_s \rightarrow V_{s+1}$ are injective.
- (2) (Surjectivity) The \mathfrak{S}_{s+1} -span of $\phi_s(V_s)$ equals all of V_{s+1} .
- (3) (Multiplicities) The decomposition of V_s into irreducible \mathfrak{S}_s -modules is of the form

$$V_s = \bigoplus_{\lambda} c_{\lambda,s} P(\lambda)_s$$

with multiplicities $0 \leq c_{\lambda,s} \leq \infty$. Moreover, for each λ , the multiplicities $c_{\lambda,s}$ are eventually independent of s .

The sequence is called **uniformly representation stable** with stable range $s \geq S$ if in addition, the multiplicities $c_{\lambda,s}$ are independent of s for all $s \geq S$ with no dependence on λ .

Let \mathbf{FI} be the category whose objects are finite sets and whose morphisms are injections. This is equivalent to the category whose objects are natural numbers s and whose morphisms $t \rightarrow s$ correspond to injections $\{1, \dots, t\} \rightarrow \{1, \dots, s\}$.

Definition 2.3. An **FI-module** (over a field \mathbb{k}) is a functor V from \mathbf{FI} to the category $\mathbf{Vect}_{\mathbb{k}}$ of vector spaces over \mathbb{k} . We denote the vector space $V(s)$ by V_s and refer to the maps $V(\phi)$ for ϕ a morphism in \mathbf{FI} as the structure maps of V .

2.1. Stability degree and weight of an FI-module. We recall the notions of stability degree and weight of an FI-module. These will imply representation stability and can be used to control the stable range of a sequence $\{V_s\}$ corresponding to an FI-module V .

Let V_s be an \mathfrak{S}_s -module. Then we denote by $(V_s)_{\mathfrak{S}_s}$ the \mathfrak{S}_s -coinvariant quotient $V_s \otimes_{\mathbb{k}\mathfrak{S}_s} \mathbb{k}$.

Definition 2.4. An FI-module V has **stability degree** t if for all $a \geq 0$, the maps $(V_{s+a})_{\mathfrak{S}_s} \rightarrow (V_{s+1+a})_{\mathfrak{S}_{s+1}}$ induced by the standard inclusions $I_s : \{1, \dots, s\} \rightarrow \{1, \dots, s+1\}$, are isomorphisms for all $s \geq t$.

We say V has **injectivity degree** \mathcal{I} (resp. **surjectivity degree** \mathcal{S}) if the maps $(V_{s+a})_{\mathfrak{S}_s} \rightarrow (V_{s+1+a})_{\mathfrak{S}_{s+1}}$ are injective $\forall s \geq \mathcal{I}$ (resp. surjective $\forall s \geq \mathcal{S}$). We say V has **stability type** $(\mathcal{I}, \mathcal{S})$.

Definition 2.5. An FI-module V has **weight** $\leq d$ if for all $s \geq 0$ every irreducible component $P(\lambda)_s$ of V_s has $|\lambda| \leq d$.

Remark 2.6. *A key property of weight is that it is preserved under subquotients and extensions. In fact, there is an alternate definition of weight: the collection of FI-modules over \mathbb{k} of weight $\leq d$ is the minimal collection which contains all FI-modules generated in degree $\leq d$ and is closed under subquotients and extensions. For more details see [2].*

Together, finite weight and stability of an FI-module V imply representation stability of the corresponding sequence of \mathfrak{S}_s -modules $\{V_s\}$. Moreover, the stability degree and weight give a measure of control on the representation stable range. The following result of Church-Ellenberg-Farb is the key to deducing representation stability of $\{H^i(\Gamma_{n,s})\}_{s \geq 0}$ from our FI-module $H^i(\Gamma_{n,\bullet})$.

Proposition 2.7 ([2], Proposition 3.3.3). *Let V be an FI-module of weight d and stability degree t . Then the sequence $\{V_s\}$ is uniformly representation stable with stable range $s \geq t + d$.*

With this in hand it is clear that Theorem A is an immediate corollary to Theorem B. Another consequence of Theorem B is the existence of stable character polynomials.

2.2. Character Polynomials. For $j \geq 1$, let $X_j : \mathfrak{S}_s \rightarrow \mathbb{N}$ be the class function defined by

$$X_j(\sigma) = \text{number of } j\text{-cycles in } \sigma.$$

A polynomial in the variables X_j is called a character polynomial. We define the degree of a character polynomial by setting $\deg(X_j) = j$. The following theorem of Church-Ellenberg-Farb says that characters of finitely generated FI-modules are eventually described by a single character polynomial, and moreover gives explicit bounds on the degree and the stable range of this polynomial in terms of weight and stability degree of the FI-module.

Theorem 2.8 ([2], Theorem 3.3.4). *Let V be a finitely generated FI-module of weight $\leq d$ and stability degree $\leq t$. There exists a unique polynomial $f_V \in \mathbb{Q}[X_1, \dots, X_d]$ of degree at most d such that for all $n \geq d + t$ and all $\sigma \in \mathfrak{S}_n$,*

$$\chi_{V_n}(\sigma) = f_V(\sigma).$$

Corollary 1.1 will thus follow immediately from Theorem B. It is worth pointing out that in particular, this shows that the dimension of $H^i(\Gamma_{n,s})$ is eventually (as s grows) given by a single character polynomial.

In [5] Conant-Hatcher-Kassabov-Vogtmann describe the \mathfrak{S}_s -module structure of $H^i(\Gamma_{n,s})$ for $n = 1, 2$, from which one can read off their irreducible \mathfrak{S}_s -module decomposition, that is, a decomposition into terms of the form $P(\lambda)_s$. We have the following classical fact, which underpins the theorem above. Fix a partition λ . There exists a unique character polynomial f_λ such that for any $s \geq |\lambda| + \lambda_1$, the character of the \mathfrak{S}_s -module $P(\lambda)_s$ is given by f_λ . In [7] they describe an algorithm constructing f_λ that we will use in conjunction with calculations from [5] to compute some explicit examples of character polynomials of various $H^i(\Gamma_{n,s})$. It is cleanest to describe these character polynomials in terms of the notation $(x)_j := x(x-1)\cdots(x-j+1)$.

Example. Fix $n = 1$ and $i = 2$. From [5], Proposition 2.7 we obtain the following decomposition of $H^2(\Gamma_{1,s})$ into irreducible \mathfrak{S}_s -modules.

$$H^2(\Gamma_{1,s}) = P\left(\begin{array}{|c|} \hline \square \\ \hline \end{array}\right)_s.$$

Using the algorithm from [7] we obtain the character polynomial $f_{\begin{array}{|c|} \hline \square \\ \hline \end{array}}$ for $P\left(\begin{array}{|c|} \hline \square \\ \hline \end{array}\right)_s$. Corollary 1.1 implies that, for $s \geq 3$, the character $\chi_{2,1}$ of $H^2(\Gamma_{1,s})$ is given by the polynomial,

$$f_{2,1}(X_1, X_2) = f_{\begin{array}{|c|} \hline \square \\ \hline \end{array}}(X_1, X_2) = \frac{1}{2} \cdot (X_1)_2 - (X_1) - (X_2) + 1.$$

We can use this, for example, to obtain that for $s \geq 3$ the dimension of $H^2(\Gamma_{1,s})$ is

$$\frac{s(s-1)}{2} - s + 1 = \binom{s-1}{2}.$$

Notice that this agrees with the description of $H^2(\Gamma_{1,s}) = \bigwedge^2 \mathbb{k}^{s-1}$ given in [5].

Example. Fix $n = 2$ and $i = 4$. From [5], Theorem 2.10 we obtain the following stable decomposition of $H^4(\Gamma_{2,s})$ into irreducible \mathfrak{S}_s -modules. For $s \geq 6$,

$$H^4(\Gamma_{2,s}) = P\left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}\right)_s \oplus P\left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}\right)_s \oplus P\left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\right)_s.$$

Using the algorithm from [7] we obtain the character polynomials $f_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}$, $f_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}}$ and $f_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}}$.

Corollary 1.1 implies that, for $s \geq 6$, the character $\chi_{4,2}$ of $H^4(\Gamma_{2,s})$ is given by the sum of these three character polynomials,

$$f_{4,2}(X_1, X_2, X_3, X_4) = \frac{1}{12}(X_1)_4 + (X_2)_2 - X_1 \cdot X_3.$$

For instance, let $\tau = (1\ 2)(3\ 4)(5\ 6 \cdots 100) \in \mathfrak{S}_{100}$. Then $\chi_{4,2}(\tau) = 2$.

Both the stable decomposition of $H^i(\Gamma_{n,s})$ and the stable character polynomials describing $\chi_{i,n}$ evident in these examples are general features of being a finitely generated FI-module. It thus remains to prove Theorem B, which we will do by analysing a spectral sequence of FI-modules. It turns out that the E_2 -page of that spectral sequence admits a particularly nice description in terms of *free FI-modules*. We recall their definition now.

2.3. Free FI-modules. Let **FI-Mod** denote the category whose objects are FI-modules over \mathbb{k} and whose morphisms are given by natural transformations. This is an abelian category in which operations like kernels, quotients, subobjects, etc., are all defined pointwise.

Let **FB** denote the category of finite sets with bijections, and similarly, let **FB-Mod** denote the category whose objects are FB-modules, i.e., functors from **FB** to $\mathbf{Vect}_{\mathbb{k}}$, and whose morphisms are natural transformations. An FB-module W determines,

and is determined by, a collection of \mathfrak{S}_n -modules W_n for each $n \in \mathbb{N}$. In particular, any \mathfrak{S}_n -module W_n determines an FB-module. There is a natural forgetful functor

$$\pi : \text{FI-Mod} \rightarrow \text{FB-Mod}$$

sending an FI-module V to the sequence of \mathfrak{S}_n -modules it determines.

Definition 2.9 ([2], Definition 2.2.2). *Define the functor*

$$M : \text{FB-Mod} \rightarrow \text{FI-Mod}$$

to be the left adjoint to π .

Explicitly $M(P_\lambda)$ takes a finite set $\mathbf{s} = \{1, \dots, s\}$ to $P_\lambda \circ P_{(s-|\lambda|)}$. We denote $M(P_\lambda)$ simply by $M(\lambda)$.

In [2] they describe the stability degree and weight of $M(\lambda)$. We stress that we are working over a field \mathbb{k} of characteristic zero, otherwise we cannot guarantee such strong bounds on surjectivity degree.

Proposition 2.10 (see [2]). *The FI-module $M(\lambda)$ has stability type $(0, \lambda_1)$ and weight at most $|\lambda|$.*

Proof. This follows from three results in [2]. The injectivity degree is given in Proposition 3.1.7, the surjectivity degree in Proposition 3.2.6 and the weight in Proposition 3.2.4. \square

Our notation for irreducible representation of \mathfrak{S}_s as $P(\lambda)_s$ is suggestive of the structure of an FI-module, and indeed this is the case.

Proposition 2.11 ([2], Proposition 3.4.1). *For any partition λ there exists an FI-module $P(\lambda)$, obtained as a sub-FI-module of the free FI-module $M(\lambda)$, satisfying*

$$P(\lambda)(\mathbf{s}) = \begin{cases} P(\lambda)_s & s \geq |\lambda| + \lambda_1 \\ 0 & \text{else} \end{cases}$$

$P(\lambda)$ has weight $\leq |\lambda|$ and stability degree $\leq \lambda_1$.

2.4. Homological techniques for FI-modules. One advantage of the FI-modules viewpoint is that it brings homological techniques to bear. In particular, the following result governs the dynamics of stability type through our spectral sequence. Recall that \mathbb{k} has characteristic 0.

Proposition 2.12 ([2], Lemma 6.3.2).

- (1) *Let U, V, W be FI-modules with stability type $(*, A), (B, C), (D, *)$ respectively, and let*

$$U \xrightarrow{f} V \xrightarrow{g} W$$

be a complex of FI-modules (i.e., $g \circ f = 0$). Then $\ker g / \text{im } f$ has injectivity degree $\leq \max(A, B)$ and surjectivity degree $\leq \max(C, D)$.

- (2) *Let V be an FI-module with a filtration*

$$0 = F_j V \subseteq F_{j-i} V \subseteq \dots \subseteq F_1 V \subseteq F_0 V = V$$

by FI-modules $F_i V$. The successive quotients $F_i V / F_{i+1} V$ have stability type $(\mathcal{I}, \mathcal{S})$ for all i if and only if V has stability type $(\mathcal{I}, \mathcal{S})$.

3. THE GROUPS $\Gamma_{n,s}$ AND THE FI-MODULES THEY DETERMINE

Let $X_{n,s}$ be a graph of rank n with s marked points $\partial := \{x_1, \dots, x_s\}$. We defined $\Gamma_{n,s}$ as the space self homotopy equivalences of $X_{n,s}$ fixing ∂ (pointwise) modulo homotopies that fix ∂ throughout. The group operation on $\Gamma_{n,s}$ is induced by composition of homotopy equivalences, which is clearly associative and admits an identity element. In [5] they prove the existence of inverse as follows. Let $f : X \rightarrow Y$ be a homotopy equivalence of graphs that sends $\partial_X = \{x_1, \dots, x_s\} \subset X$ bijectively to $\partial_Y = \{y_1, \dots, y_s\} \subset Y$. Consider the mapping cylinder of f , or rather its quotient Z obtained by collapsing the s intervals $x_i \times I$. By observing that the inclusion of Y into Z is a homotopy equivalence, and that Z deformation retracts onto X we obtain an inverse to f that acts as f^{-1} on ∂_Y as desired. Moreover, this argument shows that $\Gamma_{n,s}$ does not depend on the choice of graph $X_{n,s}$ up to isomorphism.

The proof of Theorem B in the case when $n > 1$ relies on a spectral sequence argument that itself is borne of certain short exact sequences we describe now (for full details see [5], Section 1.2).

Let $n > 1$, $s \geq 0$ and write $X = X_{n,s}$. Let E denote the space of homotopy equivalences of X with no requirement that ∂ be fixed, and let D be the space of homotopy equivalences of X that are required to fix ∂ . Thus $\Gamma_{n,s} \cong \pi_0(D)$ and $\Gamma_{n,0} = \text{Out}(F_n) \cong \pi_0(E)$. There is a map $E \rightarrow X^s$ by $f \mapsto (f(x_1), \dots, f(x_s))$ which is a fibration with fiber D over the point (x_1, \dots, x_s) . The long exact sequence of homotopy groups of this fibration ends,

$$(1) \quad \pi_1(E) \rightarrow \pi_1(X^s) \rightarrow \Gamma_{n,s} \rightarrow \text{Out}(F_n) \rightarrow 1.$$

To see that $\pi_1(E) = 1$ we consider the case $s = 1$, when (1) says,

$$\pi_1(E) \rightarrow F_n \rightarrow \text{Aut}(F_n) \rightarrow \text{Out}(F_n) \rightarrow 1.$$

The map $F_n \rightarrow \text{Aut}(F_n)$ can be seen to be conjugation, and it's kernel, $\pi_1(E)$, is thus trivial. Thus we have proved the following.

Proposition 3.1 (see [5], Proposition 1.2). *If $n > 1$ there is a short exact sequence,*

$$1 \rightarrow F_n^s \rightarrow \Gamma_{n,s} \rightarrow \text{Out}(F_n) \rightarrow 1$$

3.1. Building the FI-module $H^i(\Gamma_{n,\bullet})$. The group cohomology $H^i(\Gamma_{n,s})$ admits an action of the symmetric group \mathfrak{S}_s , and thus defines an FB-module $H^i(\Gamma_{n,\bullet})$ taking the finite set \mathbf{s} to $H^i(\Gamma_{n,s})$. We now show that $H^i(\Gamma_{n,\bullet})$ actually determines an FI-module.

Proposition 3.2. *Fix $i, n \geq 0$. $H^i(\Gamma_{n,\bullet})$ is an FI-module.*

Proof. It suffices to describe a functorial way to assign to an injection $\phi \in \text{Hom}_{\text{FI}}(\mathbf{t}, \mathbf{s})$ a linear map $H^i(\Gamma_{n,t}) \rightarrow H^i(\Gamma_{n,s})$. Fix a graph $X_{n,s}$ obtained by attaching s hairs to the rose R_n at its single vertex. The marked points are the 1-valent vertices of $X_{n,s}$, which we identify with \mathbf{s} . Define $X_{n,t}$ similarly and identify its marked points with \mathbf{t} . Pick a homotopy equivalence $f : X_{n,t} \rightarrow X_{n,s}$ that acts as ϕ on the marked points of $X_{n,t}$; that is, the marked point x in $X_{n,t}$ should be sent to the marked points $\phi(x)$ in $X_{n,s}$. Let g be a self homotopy equivalence of $X_{n,s}$ fixing the hairs so that g determines an element of $\Gamma_{n,s}$. Now the conjugate gf^{-1} is a self homotopy

equivalence of $X_{n,t}$ fixing its marked points \mathbf{t} and as such determines an element h of $\Gamma_{n,t}$.

This procedure determines a map $\Gamma_{n,s} \rightarrow \Gamma_{n,t}$ that depends on the choice of f . However, up to conjugation by $\Gamma_{n,t}$ this element h only depends on ϕ , and thus induces a well-defined map on cohomology, which doesn't see inner automorphisms. \square

Remark 3.3. *If we choose f that it only permutes the hairs of $X_{n,t}$ (i.e., induces the identity on π_1) then the map $\Gamma_{n,s} \rightarrow \Gamma_{n,t}$ can be thought of as forgetting the $s - t$ points not in the image of ϕ , and relabelling the hairs according to ϕ .*

Remark 3.4. *It is perhaps tempting to draw on the FI structure at the level of groups and try and make $\Gamma_{1,\bullet}$ a (contravariant) functor from FI to the category of groups; a co-FI-group in the language of [2]. Indeed, to an injection $\phi \in \text{Hom}_{\text{FI}}(t, s)$ we described maps from $\Gamma_{n,s}$ to $\Gamma_{n,t}$ in the proof of Proposition 3.2. However, the element fgf^{-1} in $\Gamma_{n,t}$ depended on the choice of homotopy equivalence f and as such it is false that $\Gamma_{1,\bullet}$ forms a co-FI-group in general. That being said, it is straightforward to show that $\Gamma_{1,\bullet}$ does form a co-FI-group. We have no need for that result here, and therefore don't do so.*

A useful observation is that the structure maps are always injective.

Proposition 3.5. *The structure maps of $H^i(\Gamma_{n,\bullet})$ are injective.*

Proof. It suffices to show that the maps $\Gamma_{n,s} \rightarrow \Gamma_{n,t}$ used to build the structure maps are split. In the case $t \neq 0$ this is a straightforward adaptation of a result in [5] (see Proposition 1.2) where they prove that there is a splitting $\Gamma_{n,s} \rightarrow \Gamma_{n,s-k}$ when $k < s$. The remaining case where $t = 0$ is also dealt with in [5], (see Theorem 1.4) where they prove that natural map $\text{Aut}(F_n) \rightarrow \text{Out}(F_n)$ splits on the level of (rational) (co)homology. \square

3.2. Rank 1. In rank 1 the situation is somewhat simpler and we don't need to appeal to a spectral sequence argument to witness Theorem B.

Proposition 3.6. *As FI-modules*

$$H^i(\Gamma_{1,\bullet}) = \begin{cases} P(1^i) & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

and as such $H^i(\Gamma_{1,\bullet})$ satisfies Theorem B.

Proof. The \mathfrak{S}_s -module structure of the cohomology in rank 1 was computed in [5], Proposition 2.7 to be

$$H^i(\Gamma_{1,s}) = \begin{cases} P_{(s-i, 1^i)} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

We need only consider the even case, where we have an FI-module which behaves like $P(1^i)$ when evaluated at any finite set. The fact that the irreducible decomposition contains exactly one irreducible at each finite set implies that the structure maps either agree with those of $P(1^i)$ or are zero. Proposition 3.5 says that the structure maps are injective, so we have an equality of FI-modules $H^i(\Gamma_{1,\bullet}) = P(1^i)$ when i is even. As for satisfying Theorem B, $P(1^i)$ has stability degree ≤ 1 and weight $\leq i$ by Proposition 2.11, as desired. \square

4. THE SPECTRAL SEQUENCE ARGUMENT

With the rank 1 case taken care of we proceed to prove Theorem B in higher rank by establishing a spectral sequence of FI-modules converging to $H^i(\Gamma_{n,\bullet})$. Throughout this section fix $i \geq 0$ and $n \geq 2$.

Lemma 4.1. *There is a spectral sequence of FI-modules*

$$E_2^{pq} = \bigoplus_{|\lambda|=q, \lambda_1 \leq n} C_{p,\lambda} \otimes M(\lambda) \Rightarrow_p H^i(\Gamma_{n,\bullet}),$$

converging to the FI-module $H^i(\Gamma_{n,\bullet})$, where $C_{p,\lambda}$ is a constant FI-module depending only on p and λ .

Proof. Let

$$C_{p,\lambda} = H^p(\text{Out}(F_n); \mathbb{S}_{\lambda'} H)$$

where $H := H^1(F_n) = \mathbb{k}^n$ and where $\mathbb{S}_{\lambda'}$ is the Schur functor corresponding to the conjugate partition λ' . Define E_2^{pq} as in the statement of the lemma.

We will show that, when evaluated at the finite set $\mathbf{s} = \{1, \dots, s\}$, this gives the second page of the Leray-Serre spectral sequence of groups associated to the short exact sequence

$$1 \rightarrow F_n^s \rightarrow \Gamma_{n,s} \rightarrow \text{Out}(F_n) \rightarrow 1$$

from Proposition 3.1. In other words, we will show that

$$(E_2^{pq})_s = H^p(\text{Out}(F_n); H^q(F_n^s)) \Rightarrow_p H^{p+q}(\Gamma_{n,s})$$

as a spectral sequence of groups. Functoriality of the Leray-Serre spectral sequence will complete the proof.

First observe that, by the Künneth formula, $H^q(F_n^s) = H^{\wedge q} \circ P_{(s-q)}$ as an \mathfrak{S}_s -module (this is proved carefully in [5], Lemma 2.4). We have,

$$H^p(\text{Out}(F_n); H^q(F_n^s)) = H^p(\text{Out}(F_n); H^{\wedge q} \circ P_{(s-q)}).$$

The $\text{Out}(F_n)$ action on $H^{\wedge q}$ factors through a $GL_n(\mathbb{Z})$ action. We decompose using Schur-Weyl duality giving,

$$\begin{aligned} H^p(\text{Out}(F_n); H^{\wedge q} \circ P_{(s-q)}) &= \bigoplus_{|\lambda|=q} H^p(\text{Out}(F_n); \mathbb{S}_{\lambda} H \otimes P_{\lambda'} \circ P_{(s-q)}) \\ &= \bigoplus_{|\lambda|=q} H^p(\text{Out}(F_n); \mathbb{S}_{\lambda} H) \otimes P_{\lambda'} \circ P_{(s-q)} \\ &= \bigoplus_{|\lambda|=q} H^p(\text{Out}(F_n); \mathbb{S}_{\lambda} H) \otimes M(\lambda')_s \end{aligned}$$

where λ' is the conjugate partition of λ . Now observe that $\mathbb{S}_{\lambda} H = 0$ if λ has more than n rows by the character formula (for details see [6]). Therefore λ' has at most n columns, i.e., $\lambda'_1 \leq n$. Therefore

$$H^p(\text{Out}(F_n); H^q(F_n^s)) = \bigoplus_{\substack{|\lambda'|=q \\ \lambda'_1 \leq n}} H^p(\text{Out}(F_n); \mathbb{S}_{\lambda} H) \otimes M(\lambda')_s = (E_2^{pq})_s.$$

Swapping λ with λ' completes the proof. \square

We now describe the stability type and weight of the FI-module E_2^{pq} .

Lemma 4.2. E_2^{pq} has stability type $(0, n)$ and weight q .

Proof. $C_{p,\lambda}$ are constant FI-modules and thus do not contribute to weight or stability type. $M(\lambda)$ has stability type $(0, \lambda_1)$ and weight $|\lambda|$ by Proposition 2.10. E_2^{pq} is obtained by summing over partitions $\lambda \vdash q$ with at most n columns. In particular, each λ satisfies $\lambda_1 \leq n$ and $|\lambda| = q$. \square

We are ready to give the spectral sequence argument.

Lemma 4.3. The FI-modules E_k^{pq} on the k th page of the spectral sequence have stability degree n .

Proof. We denote the stability type of E_k^{pq} by $(\mathcal{I}_k^{pq}, \mathcal{S}_k^{pq})$. We use Lemma 2.12 and the fact that the spectral sequence is concentrated in the first quadrant to inductively produce bounds on stability type in subsequent pages.

On the 2nd page all terms E_2^{pq} have stability type at most $(0, n)$. To compute the terms on the third page we use the differentials d_2^{pq} of bidegree $(2, -1)$. We indicate the stability type of terms for convenience.

$$\begin{array}{ccccc} E_2^{p-2, q+1} & \xrightarrow{d_2^{p+2, q-1}} & E_2^{pq} & \xrightarrow{d_2^{pq}} & E_2^{p+2, q-1} \\ (0, \mathcal{S}_2^{p-2, q+1}) & & (0, n) & & (0, n) \end{array}$$

where

$$\mathcal{S}_2^{p-2, q+1} = \begin{cases} 0 & p = 0, 1 \\ n & p \geq 2 \end{cases}$$

depending on whether or not $E_2^{p-2, q+1}$ is in the first quadrant. Now Lemma 2.12 gives us that

$$\begin{aligned} \mathcal{I}_3^{pq} &= \max(\mathcal{S}_2^{p-2, q+1}, \mathcal{I}_2^{pq}) = \begin{cases} 0 & p = 0, 1 \\ n & p \geq 2 \end{cases} \\ \mathcal{S}_3^{pq} &= \max(\mathcal{S}_2^{pq}, \mathcal{I}_2^{p+2, q-1}) = n. \end{aligned}$$

We proceed similarly with the inductive step. We have

$$\begin{array}{ccccc} E_k^{p-k, q+k-1} & \xrightarrow{d_k^{p+k, q-k+1}} & E_k^{pq} & \xrightarrow{d_k^{pq}} & E_k^{p+k, q-k+1} \\ (*, \mathcal{S}_k^{p-k, q+k-1}) & & (\mathcal{I}_k^{pq}, n) & & (n, *) \end{array}$$

and can finally conclude that

$$\begin{aligned} \mathcal{I}_{k+1}^{pq} &= \max(\mathcal{S}_k^{p-k, q+k-1}, \mathcal{I}_k^{pq}) = \begin{cases} 0 & p = 0, 1 \\ n & p \geq 2 \end{cases} \\ \mathcal{S}_{k+1}^{pq} &= \max(\mathcal{S}_k^{pq}, \mathcal{I}_k^{p+k, q-k+1}) = n, \end{aligned}$$

since $\mathcal{S}_k^{p-k, q+k-1} = n$ unless $p < k$ when it is 0, and $\mathcal{I}_k^{pq} = n$ unless $p = 0, 1$ when it is 0. \square

We are now ready to prove our main result.

Proof of Theorem B. First observe that $E_\infty^{p,i-p}$ has weight $\leq i$ and stability degree $\leq n$. Indeed $E_\infty^{p,i-p}$ is a subquotient of $E_2^{p,i-p}$ and thus has weight $\leq i - p \leq i$ by Lemma 4.2, and $E_\infty^{p,i-p}$ has stability degree n by Lemma 4.3.

We have that E_2^{pq} is a first quadrant spectral sequence of FI-modules converging to the FI-module $H^{p+q}(\Gamma_{n,\bullet})$. This tells us that there exists a natural filtration of $H^i(\Gamma_{n,\bullet})$ whose graded quotients are $E_\infty^{p,i-p}$ and so, by Proposition 2.12, $H^i(\Gamma_{n,\bullet})$ has stability degree n . Moreover, since weight is preserved under extensions, $H^i(\Gamma_{n,\bullet})$ has weight $\leq i$. \square

Remark 4.4. In [10] Jiménez Rolland develops a general framework for dealing with spectral sequences of FI-modules. In particular, the description given in the proof of Lemma 4.1 shows that $H^q(F_n^s)$ has weight and stability degree $\leq q$. Thus, in the notation of [10] Theorem 5.3 we have shown that $\beta = 1$ and that $H^i(\Gamma_{n,\bullet})$ has weight $\leq i$ and stability type $(2i, i)$. We note that this recovers the representation stability bounds of [5] upon which we just improved.

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